

Rigid-plastic large-deformation analysis of geotechnical penetration problems

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Abstract

A new sequential limit analysis scheme with application to geotechnical penetration problems is presented. The new scheme relies on previously developed procedures for classical limit analysis as well as on new adaptive remeshing techniques and robust and efficient mathematical programming algorithms. Two plane strain problems involving large penetration are presented to demonstrate the capabilities of the new scheme.

1 INTRODUCTION

Geotechnical penetration problems are quite common and include the installation of piles and mandrels, cone penetration testing as well as various applications involving earth-burrowing missiles ('bunker busters'). Common to these problems are the extreme deformations induced by the penetrating object. From a numerical point of view, such scenarios call for specialized methods. So far, the line of attack has been to extend the usual infinitesimal-deformation finite element formulations into the finite-deformation regime. Examples include the works of Hu & Randolph (1998), Sheng et al. (2009), and Nazem et al. (2009). In principle, such methods are capable of accounting for the exact physical behaviour provided, of course, that appropriate constitutive models are available. Such models are usually formulated and calibrated in the infinitesimal-deformation while the finite-deformation range remains largely unexplored. In addition, the 'complete' approach involves a number of other serious complications including severe mesh distortion and mapping of state variable from one position to another in the course of the penetration.

These uncertainties with the material behaviour and complications with practical implementation motivate a re-examination of the perhaps oldest approach to large-deformation analysis of materials displaying a plastic behaviour, namely that of limit analysis, i.e. assuming a rigid-perfectly-plastic material behaviour. With this assumption and an estimate of the geometry – original as well as deformed – the problem reduces to one identical to standard bearing capacity. More specifically the idea is to perform a sequence of limit analysis with

each new configuration being updated on the basis of the previous failure mode. This scheme, in the following denoted *sequential limit analysis*, can, as conventional limit analysis, be carried out either analytically or numerically. Recent analytical studies include those of Maciejewski and Jarzebowski (2004) who considered a large-deformation version of the classic passive earth pressure problem, Maillot & Leroy (2006) who considered the problem of kink folding in geological structures, and Hambleton & Drescher (2010) who solve various problems of indentation in cohesive-frictional materials. It is further worth noting that sequential limit analysis has a long history in metal plasticity where problems such as drawing, rolling and stamping have been considered within this framework (see e.g. Hill 1950, Kachanov 1971). Numerical formulations of sequential limit analysis have also mostly been confined to metals. Applications include sheet metal forming (Raithatha & Duncan 2009), extrusion problems (Leu 2005), and steel structures (Kim & Hu 2006).

In the following, a numerical formulation of sequential limit analysis, with particular emphasis on geotechnical penetration problems, is presented. In contrast to previous schemes in the field, the mesh distortions are of such a severity that remeshing must be considered. In addition to describing the new geometry in each time step, the remeshing is adaptive in the sense that the mesh is refined with an aim of improving the limit loads computed in each time step. Two examples demonstrating the capabilities of the new scheme are presented. These both involve plane strain deep penetration problems which in conventional finite element formulations are known to be particularly difficult.

Regarding the simplified material behaviour assumed, it should be noted that since the larger part of the strains accumulated in most finite deformation problems can be expected to be of the plastic kind. As such, the assumption of a rigid-perfectly plastic material behaviour is not altogether unreasonable. This is true for metals and perhaps even more so for soils where the concept of elasticity in many ways is questionable. However, more complex models are entirely possible to treat within the general framework presented. Indeed, common models involving nonlinear elasticity, hardening yield surfaces, viscoplasticity, etc can be included without any noteworthy complications following the work of Krabbenhoft et al. (2005), Krabbenhoft et al. (2007), Krabbenhoft (2009) in the infinitesimal-deformation regime.

2 LIMIT ANALYSIS

2.1 Classical Limit Analysis

Classical limit analysis is based on a rigid-perfectly-plastic material behaviour and further assumes that the deformations up to the point of collapse are small. Consider a body of such a material occupying a V with boundary S . The governing equations are here given by:

$$\tilde{\mathbf{V}}^T \boldsymbol{\sigma} + \mathbf{b}_0 = \mathbf{0} \quad \text{in } V \quad (1)$$

$$\mathbf{N}^T \boldsymbol{\sigma} = \alpha \mathbf{t}_0 \quad \text{on } S$$

$$\tilde{\mathbf{V}} \dot{\mathbf{u}} = \dot{\lambda} \nabla_{\sigma} F(\boldsymbol{\sigma}) \quad (2)$$

$$F(\boldsymbol{\sigma}) \leq 0, \dot{\lambda} \geq 0, \dot{\lambda} F(\boldsymbol{\sigma}) = 0 \quad (3)$$

where $\boldsymbol{\sigma}$ are the stresses, $\mathbf{u}$ are the displacements, and $\dot{\lambda}$ is the plastic multiplier. As usual, a superposed dot indicates differentiation with respect to pseudo-time (consequently, $\dot{\mathbf{u}}$ are referred to as the velocities), $\tilde{\mathbf{V}}$ is the standard linear strain-displacement operator (its transpose being the equilibrium operator), and $\mathbf{N}^T = \tilde{\mathbf{V}}^T(\mathbf{n}\mathbf{x}^T)$ with $\mathbf{n}$ being the outward normal to the boundary and $\mathbf{x}$ the spatial coordinate. The loads are given by a set of predefined body forces, $\mathbf{b}_0$, and a set of predefined tractions, $\mathbf{t}_0$, which are magnified by the collapse load factor α . Finally, the yield function is given by $F(\boldsymbol{\sigma})$. The task is now to determine α such that the equilibrium equations and static boundary conditions (1), the stress-strain relations incorporation the associated flow rule (2), and the yield and complementarity conditions (3) are satisfied. In addition, in order to limit the magnitude of the external rate of work, a condition of the type:

$$\int_S \mathbf{t}_0^T \dot{\mathbf{u}} dS = 1 \quad (4)$$

is usually enforced.

The above equations may be cast in terms of a variational statement, namely (Krabbenhoft et al. 2005, Krabbenhoft 2009):

$$\begin{aligned} \max_{\sigma, \alpha} \min_{\dot{\mathbf{u}}} \quad & \alpha + \int_V \boldsymbol{\sigma}^T \tilde{\mathbf{V}} \dot{\mathbf{u}} dV - \int_V \mathbf{b}_0^T \tilde{\mathbf{V}} \dot{\mathbf{u}} dV - \alpha \int_S \mathbf{t}_0^T \dot{\mathbf{u}} dS \\ \text{subject to} \quad & F(\boldsymbol{\sigma}) \leq 0 \end{aligned} \quad (5)$$

The max-part of this principle constitutes von Mises' principle of maximum plastic dissipation while the min-part concerns the total potential energy and corresponds to enforcing equilibrium. The upper and lower bound theorems follow as special cases of the principle.

2.1.1 Finite Element Discretization

The principle (5) is readily discretized in terms of finite elements, namely by postulating approximations to the state variables $\boldsymbol{\sigma}$ and $\mathbf{u}$. Using standard finite element notation, we have

$$\boldsymbol{\sigma} \approx \mathbf{N}_{\sigma}(\mathbf{x}) \hat{\boldsymbol{\sigma}} \quad (6)$$

$$\mathbf{u} \approx \mathbf{N}_u(\mathbf{x}) \hat{\mathbf{u}}, \quad \tilde{\mathbf{V}} \dot{\mathbf{u}} \approx \mathbf{B}_u(\mathbf{x}) \dot{\hat{\mathbf{u}}}$$

where $\mathbf{B}_u(\mathbf{x}) = \tilde{\mathbf{V}} \mathbf{N}_u(\mathbf{x})$. In standard finite formulations, the stress shape functions in $\mathbf{N}_{\sigma}$ are chosen as being continuous within the elements and discontinuous between elements while $\mathbf{N}_u$ in addition are continuous between elements and usually one polynomial degree higher than $\mathbf{N}_{\sigma}$. Such formulations may under certain circumstances result in rigorous upper bounds (Krabbenhoft 2007). Conversely, it is possible to interchange the approximation properties of the stresses and the displacements such that the approximate stress distributions are assumed continuous between elements while the displacements are discontinuous and of lower polynomial degree than the stresses. Under certain circumstances, this choice of finite element approximation leads to a rigorous lower bound formulation. Substituting the above approximations into the variational principle (5) leads to the following discrete principle

$$\begin{aligned} \max_{\sigma, \alpha} \min_{\dot{\mathbf{u}}} \quad & \alpha + \boldsymbol{\sigma}^T \mathbf{B} \dot{\hat{\mathbf{u}}} - \mathbf{p}_0^T \dot{\hat{\mathbf{u}}} - \alpha \mathbf{p}^T \dot{\hat{\mathbf{u}}} \\ \text{subject to} \quad & F(\boldsymbol{\sigma}) \leq 0 \end{aligned} \quad (7)$$

where

$$\mathbf{B} = \int_V \mathbf{N}_{\sigma}^T \mathbf{B}_u \dot{\hat{\mathbf{u}}} dV, \quad \mathbf{p}_0 = \int_V \mathbf{N}_u^T \mathbf{b}_0 dV, \quad \mathbf{p} = \int_S \mathbf{N}_u^T \mathbf{t}_0 dS \quad (8)$$

end where superposed hats to indicated discrete quantities has been dropped. Solving the min-part of (7) first yields the following problem:

$$\begin{aligned} \max_{\sigma, \alpha} \quad & \alpha \\ \text{subject to} \quad & \mathbf{B}^T \boldsymbol{\sigma} = \alpha \mathbf{p} + \mathbf{p}_0 \\ & F(\boldsymbol{\sigma}) \leq 0 \end{aligned} \quad (9)$$

which is of the familiar lower bound type (maximize the collapse load multiplier subject to yield and equilibrium-type conditions) although upper bounds on in fact may result depending on the particular finite element discretization as discussed above. Indeed, in the following, we have used such an upper bound formulation. The displacements are here interpolated quadratically over six-node triangles while the stresses vary linearly within each triangle.

2.1.2 Optimization Algorithms

The finite-dimensional optimization problem (9) may be solved in a number of ways using both general (e.g. Lyamin and Sloan 2002a,b, Krabbenhoft and Damkilde 2003) or more specialized methods. A relatively recent approach is to cast the problem in terms of a so-called conic program. Such programs can be solved very efficiently by means of dedicated algorithms. Examples include the second-order cone programming (SOCP) solver MOSEK of Andersen et al. (2003) and the more general conic programming solver SeDuMi originally due to Sturm (1999). The Authors have recently developed a new solver, SONIC, along the lines of Andersen et al. (2003) which has been used to solve the problems in the present paper. A Windows executable of the solver is available from the corresponding author. SONIC accepts problems in the SOCP standard form:

$$\begin{aligned} &\text{minimize} && \mathbf{c}^T \mathbf{x} \\ &\text{subject to} && \mathbf{Ax} = \mathbf{b} \\ &&& \mathbf{x} \in K \end{aligned} \quad (10)$$

where K defines a second-order cone:

$$K : \left\{ \mathbf{x} \in \mathbb{R}^n \mid x_1 \geq (x_2^2 + \dots + x_n^2)^{\frac{1}{2}} \right\} \quad (11)$$

A number of relevant yield criteria can be cast in this form including the Drucker-Prager criterion as well as the plane strain versions of the Tresca and Mohr-Coulomb criteria (Krabbenhoft et al. 2007). The full three-dimensional Mohr-Coulomb criterion can be cast in terms of a set of semidefinite constraints as shown by Krabbenhoft et al. (2008).

As part of the solution of (9), the dual variables are recovered. It may be shown that the dual variables, or Lagrange multipliers, associated with the equilibrium type constraints are the velocities which thus are recovered even though the problem is cast in terms of stress variables. This is a useful property, not least in connection with sequential limit analysis.

2.2 Sequential Limit Analysis

As previously stated, the basic idea of any sequential limit analysis scheme is to perform a series of successive standard limit analysis computations where

the geometry of the problem at each pseudo time step is updated based on the velocity field information of the previous analysis collapse mechanism.

Hence, at the end of each step, the algorithm for the geometry update starts by identifying the mesh boundary segments, setting the limits for regions with distinct material properties. Then, the coordinates of the nodes defining these segments are displaced accordingly to the current velocity field and the adopted pseudo-time increment. Due to lack of better information, the velocities are assumed as constant during each time step.

The size of the pseudo-time increments is stipulated in order to achieve the desired maximum nodal displacement, a prescribed parameter defined at the beginning of the analysis.

Moreover, any inconsistent overlapping of the elements in the updated configuration of the existing mesh must be avoided so that the boundary segments can be properly identified. This can be accomplished by reducing the pseudo-time size increment whenever an element in its deformed configuration is left with a negative Jacobian determinant, in other words, a *negative area*.

Once defined both the current size step and the required boundary segment data, the remaining information of the existing mesh can simply be destroyed.

Mention must be made for the fact that, in contrast with other more conventional large-deformation approaches, in the sequential limit analysis there is no need to perform any kind of mapping between internal state variables of the previous iteration mesh and the new one, due to the fact that the collapse load is not affected by an initial stress/strain state (Lubliner 1990). For this reason, any issue related with mesh distortion is non-existent in this kind of approach.

Lastly, and to conclude the geometry update algorithm, adjacent regions are tested to verify if there is any interpenetration between them in the updated configuration. If so, a new contact interface is defined and the overlapping material is discarded.

At this stage we are able to proceed with the limit analysis of the current problem configuration. An initial coarse mesh is generated over the existing regions, which, during the analysis, is gradually refined following the adaptive mesh refinement strategy proposed in (Lyamin et al. 2004).

3 EXAMPLES

3.1 Strip Footing

The first example concerns the strip footing shown in Figure 2. The footing is 1 m wide, sits on a weightless, purely cohesive soil and is subjected to a monotonically increasing central vertical load. This example has been used by Nazem et al. (2006, 2010),

to test conventional large-deformation elastoplastic analyses. The small-deformation limit load is given by

$$V/B = (2 + \pi)c_u \approx 5.14c_u \quad (12)$$

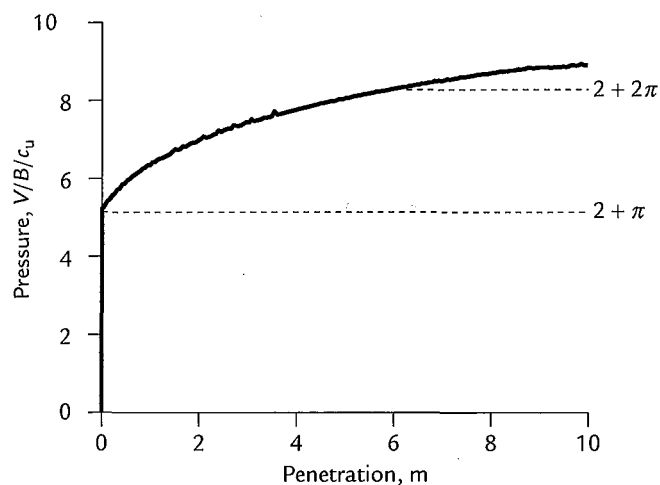


Figure 1 Load-deformation curve for strip footing on weightless soil.

Moreover, for a strip footing with a sufficiently large embedment, the mode of failure changes from a surface-type mechanism to a local mechanism with a limit load given by

$$V/B = (2 + 2\pi)c_u \approx 8.28c_u \quad (13)$$

Hence, for the complete large-deformation problem it might be expected that the load would tend to (12) as the deformations induced would render the geometry equivalent to that of a strip footing with a large embedment. Indeed, this was the conclusion of Nazem et al. (2009). However, the results of the present analysis are, somewhat different. Thus, from the load-deformation curve shown in Figure 1, we see that the load keeps increasing beyond $V/B = (2 + 2\pi)c_u \approx 8.28c_u$. The explanation for this can be found in the deformation patterns shown in Figure 2. Thus, when the local mechanism becomes active [Figure 2(d)], the soil collapses into the cavity and the geometry of the problem changes slightly

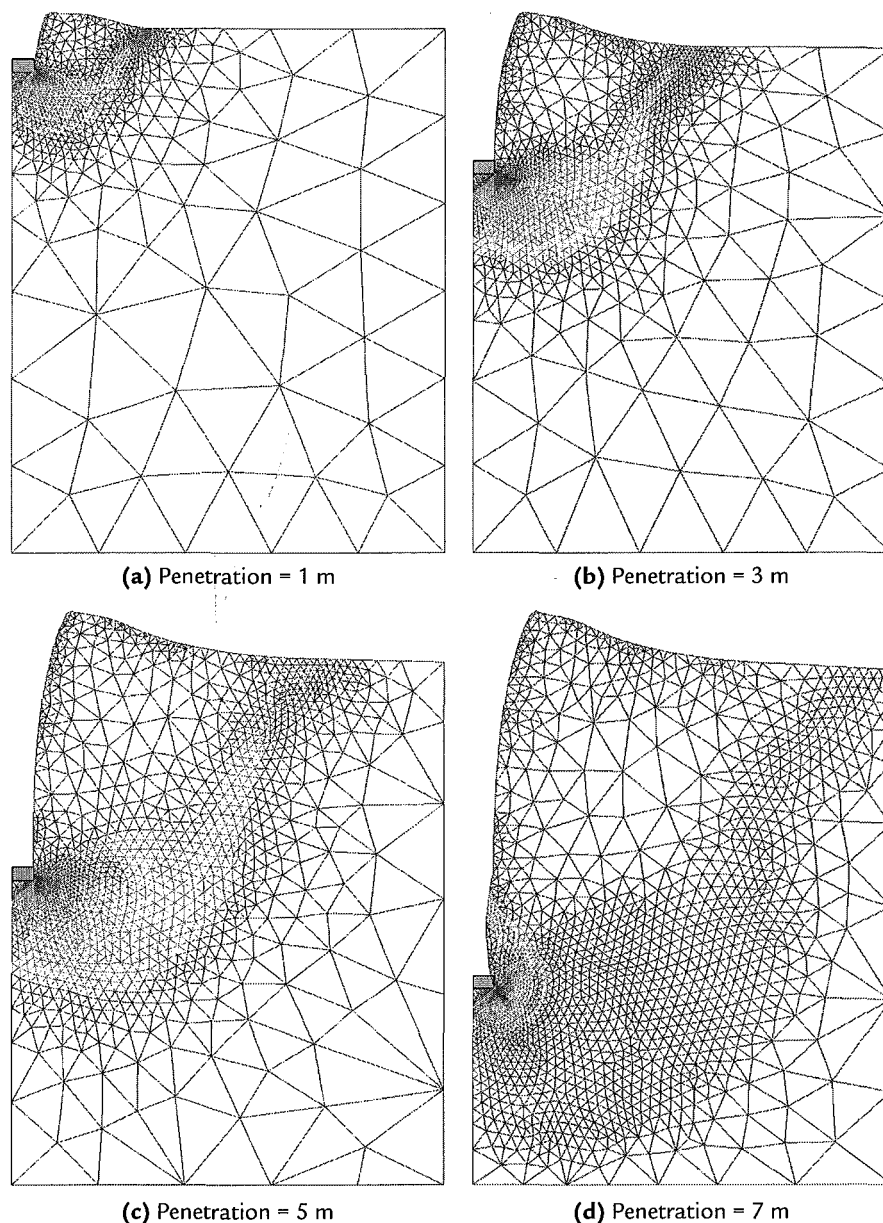


Figure 2 Penetration of strip footing of half-width $B/2 = 0.5$ m in a weightless, purely cohesive soil.

from that assumed in the large-embedment small deformation problem and the corresponding load is somewhat larger.

3.2 Wall

The next example concerns the penetration of a wall, (i.e. the plane strain analogue of a pile) into a ponderable, purely cohesive soil (see Figure 3). In conventional finite element formulations this problem

is considered particularly difficult and special techniques are often used to ensure that the rigid object penetrates the soil properly. These include the so-called “zipper technique” where the mesh gradually is “unzipped” as the pile penetrates the soil. In the present approach, such techniques are unnecessary. Indeed, the effects of and the progressive penetration are fully accounted for without resorting to any other technique than the aforementioned adaptive

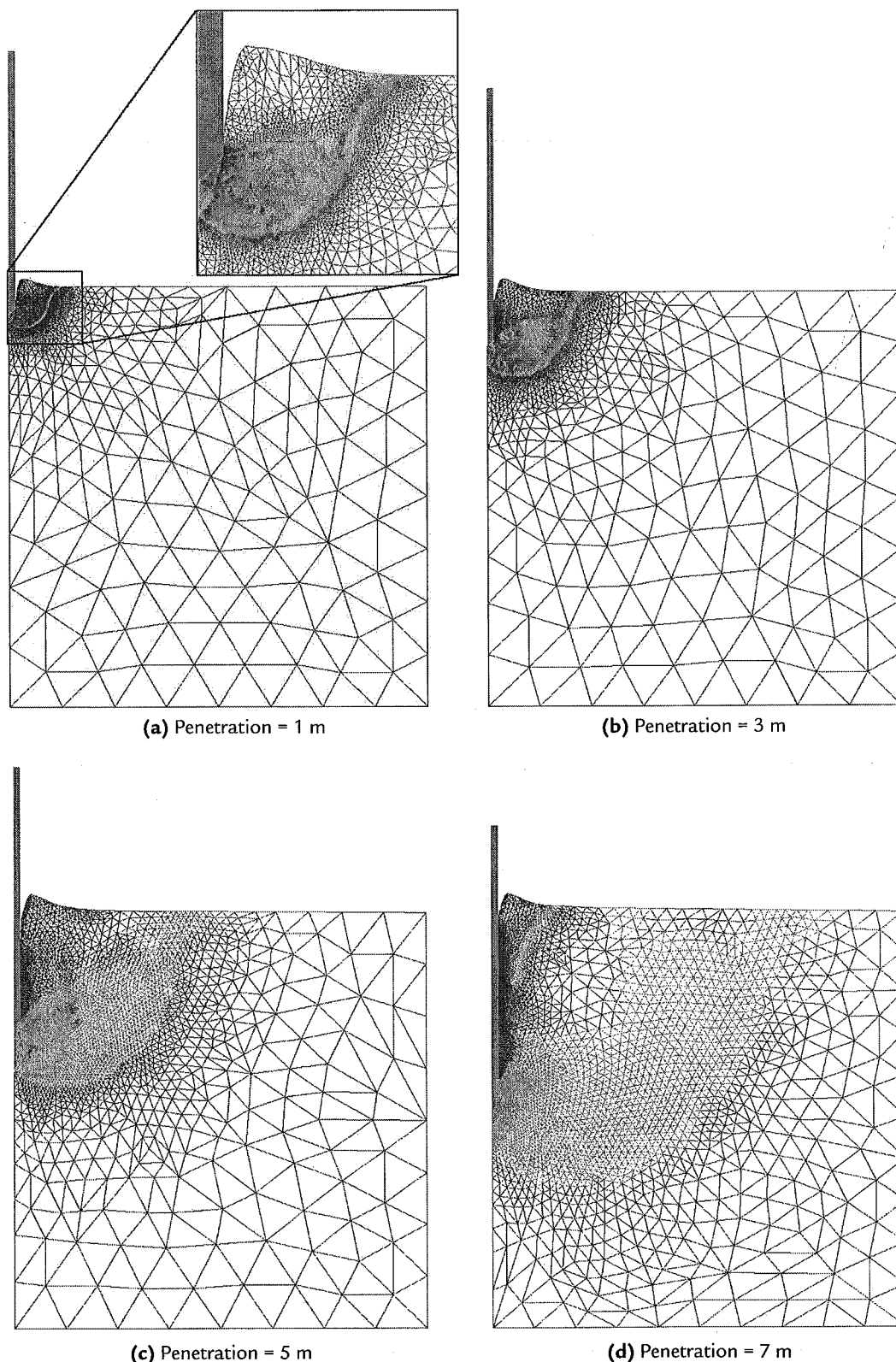


Figure 3 Penetration of wall (plane strain equivalent of pile) of half-width $B/2 = 0.5$ m. The tip inclination is 45 degrees from horizontal. Soil parameters: $c_u = 20$ kPa, $\gamma_{\text{sat}} = 20$ kN/m³.

mesh refinement, an example of which is shown in Figure 3. From this figure we also see that relatively large areas at a fair distance away from the pile contribute significantly to the overall behaviour. In other words, the plastic zone is not confined to a narrow neighborhood around the pile as is often assumed in or deducted from conventional finite element analyses.

4 CONCLUSIONS

A sequential limit analysis formulation has been presented and two examples given to demonstrate its capabilities. Future enhancements will focus on extending the methodology to more realistic soil models following the general variational framework of Krabbenhoft et al. (2005, 2009).

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